$\text{SAut}(F_5)$ HAS PROPERTY (T)

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Outline

Group Laplacians & Property (T)
Positivity & SDP
The procedure
Concrete examples
- $G = \langle S \mid R \rangle$ is a finitely presented group generated by a fixed symmetric generating set (i.e. $S^{-1} = S$).
GROUP LAPLACIANS
Group Laplacian $\Delta$ 

**Definition**

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- the operator $\Delta$ is $*$-invariant,
- spectrum of $\Delta$ is real and non-negative;
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- the operator \( \Delta \) is \(*\)-invariant,
- spectrum of \( \Delta \) is real and non-negative;
- the second eigenvalue \( \lambda_1 \) is called the \textbf{spectral gap}

\[ 0 = \lambda_0 \leq \lambda_1 \leq \cdots \]
For an orthogonal representation $\pi: G \to B(H)$ of $G$ on a (real) Hilbert space $H$ denote by $H_\pi = \{ v \in H : \pi(g)v = v \text{ for all } g \in G \}$ the (closed) subspace of $\pi$-invariant vectors.

We define $\kappa(G, S, \pi) = \inf_{\|\xi\|=1} \{ \sup_{g \in S} \| \pi(g)\xi - \xi \|_H : \xi \in (H_\pi)^\perp \}$. Definition The Kazhdan's constant $\kappa(G, S)$ is defined as $\kappa(G, S) = \inf_{\pi} \kappa(G, S, \pi)$ over all orthogonal representations $\pi$ of $G$. We say that $G$ has the Kazhdan's property (T) if and only if there exists a (finite) generating set $S$ such that $\kappa(G, S) > 0$. 
For an orthogonal representation $\pi : G \to B(\mathcal{H})$ of $G$ on a (real) Hilbert space $\mathcal{H}$ denote by

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- Provide a constructive (computable) proof for $n = 5$. 
Random group elements in finite groups:
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**Theorem (Lubotzky & Pak, 2000)**

Let $K$ be a finite group generated by $k \leq n$ elements. If $\text{SAut}(F_n)$ has property (T) with constant $\kappa = \kappa(\text{SAut}(F_n), \{\text{transvections}\}) > 0$, then PRA walk on $\Gamma_n = \Gamma_n(K)$ has fast mixing rate, i.e.

$$\|Q_t(g) - U\|_{tv} \leq \epsilon$$

for $t \leq 16\frac{\kappa^2 \log |\Gamma_n|}{\epsilon}$.

Note: We do observe fast mixing rate in practice for large $n$. 
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($Q_{(g)}$ is a random walk on the graph $\Gamma_n$ starting at generating $n$-tuple $(g)$).
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Connection to eigenvalues of $\Delta$: $\kappa(G, S)$ can be estimated as

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**Corollary**

Let \( G = \langle S | \ldots \rangle \) be a finitely generated group. If there exists \( \lambda > 0 \) such that \( \Delta^2 - \lambda \Delta \geq 0 \), then \( G \) has property (T) with

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How to prove that $\Delta^2 - \lambda \Delta \geq 0$?
How to prove that a polynomial $f \geq 0$?
Positivity & SDP
Hilbert’s 17th problem

**Theorem (Hilbert’s Positivstellensatz, 1888)**

An everywhere non-negative polynomial $p \in \Sigma^2 \mathbb{R}[x_1, \ldots, x_n]$ (is a sum of squares) if and only if either

- $n = 1$, or
- $n = 2$ and $\deg p = 2, 4$. 

Example (Motzkin, 1970s)

$$x^4 y^2 + x^2 y^2 - 3 x^2 y^2 + 1 \geq 0$$

but not SOS.

Theorem (Artin, 1924)

$p \geq 0 \iff \exists q: q^2 p \in \Sigma^2 \mathbb{R}[x_1, \ldots, x_n]$ (i.e. $p$ is a sum of squares of rational functions).

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we can obtain any polynomial of degree 2 in \( x \) and \( y \) with coefficients linear functions of \( p_{ij} \).
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For \( f \) to be a SOS we just need \((p_{ij}) = P\) to be **semi-positive definite**, 

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For \( f \) to be a SOS we just need \((p_{ij}) = P\) to be **semi-positive definite**, since then \( P = Q^T Q \) and (for \((1, x, y)^T = \vec{x}\))

\[
f = \vec{x}^T P \vec{x} = \vec{x}^T Q^T \cdot Q \vec{x} = (Q\vec{x})^T \cdot (Q\vec{x})
\]
Mathematical Programming

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- optimise linear functional
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**Linear programming:**
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**Semi-definite programming**
- optimise linear functional
- on a polytope intersected with the cone of SPD matrices (spectrahedron)
- weak duality, non-unique solutions
- even feasibility is a hard problem!
variables: $a, b_1, b_2, c, \lambda$. 
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**Example (SDP problem)**

maximise: $\lambda$

subject to: $\begin{bmatrix} c & b_2 \\ b_1 & a \end{bmatrix} \succeq 0$

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tries to maximise $\lambda$ as long as $(2x^2 + 4x + 1) - \lambda \geq 0$. 
Theorem (Schmüdgen) For any $\ast$-invariant element $\xi \in R[G]$, $\xi \geq 0 \iff \pi(\xi) \geq 0$ in every $\ast$-representation of $R[G]$.

Example $\Delta_2$ is an interior point of $\Sigma_2 R[G]$, i.e. to show that $\Delta_2 - \lambda \Delta_\ast \geq 0$ it's enough to answer if $\Delta_2 - \lambda \Delta_\ast + \epsilon \in \Sigma_2 R[G]$ for all $\epsilon > 0$, where $\lambda$ is an interior point of $\Sigma_2 R[G]$.

This of no use for us: SOS decompositions $\Delta_2 - \lambda \Delta_\ast + \epsilon = \sum \xi^\ast \xi$ may be very different for different $\epsilon$. 
Theorem (Schmüdgen)

For any $\ast$-invariant element $\xi \in R[G]$, $\xi \preceq 0 \iff \xi + \epsilon u \in \Sigma^2 R[G]$ for all $\epsilon > 0$, where $u$ is an interior point of $\Sigma^2 R[G]$.

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$\Delta^2 - \lambda \Delta \preceq 0$ is enough to answer Is $\Delta^2 - \lambda \Delta + \epsilon \in \Sigma^2 R[G]$ for all $\epsilon$? This of no use for us: SOS decompositions $\Delta^2 - \lambda \Delta + \epsilon = \sum \xi_i^* \xi_i$ may be very different for different $\epsilon$. 
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NC-Positivstellensatz
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$\Delta$ is an interior point of $\Sigma^2 I[G] = I[G] \cap \Sigma^2 \mathbb{R}[G]$
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$\Delta$ is an interior point of $\Sigma^2 I[G] = I[G] \cap \Sigma^2 \mathbb{R}[G]$, i.e. for a $\ast$-invariant element $\xi \in I[G]$

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for all $\epsilon > 0$. 

**Example**

If we can show that $\Delta^2 - \lambda \Delta + \epsilon_0 \Delta = \sum \xi_i \xi_i$ for a single fixed $\epsilon_0$, then

$$\Delta^2 - (\lambda - \epsilon_0) \Delta + \epsilon \Delta = \sum \xi_i \xi_i + \epsilon \sum (1 - g) \ast (1 - g) \in \Sigma^2 I[G]$$

for all $\epsilon$ simultaneously!
Proposition (Ozawa, 2015)

\( \Delta \) is an interior point of \( \Sigma^2 I[G] = I[G] \cap \Sigma^2 \mathbb{R}[G] \), i.e.
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\[ \Delta^2 - (\lambda - \varepsilon_0) \Delta + \varepsilon \Delta = \sum \xi_i^* \xi_i + \varepsilon \sum (1 - g)^* (1 - g) \in \Sigma^2 I[G] \]
**Proposition (Ozawa, 2015)**

$\Delta$ is an interior point of $\Sigma^2 I[G] = I[G] \cap \Sigma^2 \mathbb{R}[G]$, i.e.

for a $\ast$-invariant element $\xi \in I[G]$

$$
\xi \geq 0 \iff \xi + \varepsilon \Delta \in \Sigma^2 I[G]
$$

for all $\varepsilon > 0$.

**Example**

If we can show that $\Delta^2 - \lambda \Delta + \varepsilon_0 \Delta = \sum \xi_i^* \xi_i$ for a single fixed $\varepsilon_0$, then

$$
\Delta^2 - (\lambda - \varepsilon_0) \Delta + \varepsilon \Delta = \sum \xi_i^* \xi_i + \varepsilon \sum (1 - g)^* (1 - g) \in \Sigma^2 I[G]
$$

for all $\varepsilon$ simultaneously!
1. Pick \( G = \langle S | R \rangle \);
2. Set \( \vec{x} = (e, g_1, g_2, \ldots, g_n), g_i \in B_2(d, e, S) \) \((d = 2, 3, \ldots)\);
3. Solve the problem (numerically):
   \[
   \max \lambda \text{ subject to: } \sum P \leq 0, P \in M \vec{x} \lambda \leq 0 (\Delta_2 - \lambda \Delta_1) t = \sum g_i - 1 h, \text{ for all } t \in B_2(d, e, S) ;
   \]
4. Compute \( \sqrt{P} = Q = \begin{bmatrix} q_e, \ldots, q_{g_n} \end{bmatrix} \);
5. Finally:
   \( \xi g = \langle \vec{x}, q_g \rangle \) and \( \Delta_2 - \lambda \Delta_1 = \sum g_i \xi g^* \xi g \).
1. Pick $G = \langle S|\mathcal{R} \rangle$;
Action Plan

1. Pick $G = \langle S|\mathcal{R}\rangle$;
2. Set $\vec{x} = (e, g_1, g_2, \ldots, g_n), \quad g_i \in B_d(e, S) \ (d = 2, 3, \ldots)$;
1. Pick $G = \langle S|\mathcal{R} \rangle$;

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3. Solve the problem (numerically):

   maximize: $\lambda$

   subject to: $P \succeq 0, \quad P \in \mathbb{M}_\mathbf{x}$

   $\lambda \geq 0$

   $(\Delta^2 - \lambda \Delta)_t = \sum_{g^{-1}h=t} P_{g,h}, \quad \text{for all } t \in B_{2d}(e, S)$
1. Pick $G = \langle S|R \rangle$;
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   $\lambda \geq 0$
   
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4. Compute $\sqrt{P} = Q = [\vec{q}_e, \ldots, \vec{q}_{gn}]$
1. Pick $G = \langle S|\mathcal{R}\rangle$;

2. Set $\mathbf{x} = (e, g_1, g_2, \ldots, g_n)$, $g_i \in B_d(e, S)$ ($d = 2, 3, \ldots$);

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   subject to: $P \succeq 0$, $P \in \mathbb{M}_\mathbf{x}$
   
   $\lambda \geq 0$
   
   $(\Delta^2 - \lambda \Delta)_t = \sum_{g^{-1}h=t} P_{g,h}$, for all $t \in B_{2d}(e, S)$

4. Compute $\sqrt{P} = Q = [\mathbf{q}_e, \ldots, \mathbf{q}_g]$

5. Finally: $\xi_g = \langle \mathbf{x}, \mathbf{q}_g \rangle$ and $\Delta^2 - \lambda \Delta = \sum_{g \in \mathbf{x}} \xi_g^* \xi_g$. 
How do we certify that the numerical result is sound?
## Lemma (Netzer&Thom)

Let \( r \in I[G] \subseteq \mathbb{R}[G] \) such that \( \text{supp}(r) \subseteq B_d(e) \). Then

\[
    r + 2^{d-1}\|r\|_1 \cdot \Delta \in \Sigma^2 I[G].
\]

## Corollary

If \( \Delta^2 - \lambda \Delta = \sum \xi^*_i \xi_i + r \), then

\[
    \Delta^2 - \left( \lambda - 2^{d-1}\|r\|_1 \right) \Delta = \sum \xi^*_i \xi_i + \left( r + 2^{d-1}\|r\|_1 \Delta \right) \in \Sigma^2 I[G],
\]

i.e. \( \Delta \) has spectral gap of at least \( \lambda - 2^{d-1}\|r\|_1 \).
1. Pick $G = \langle S | R \rangle$;
2. Set $\vec{x} = (e, g_1, g_2, \ldots, g_n)$, $g_i \in B_d(e, S)$ ($d = 2, 3, \ldots$);
3. Solve the problem (numerically):
   
   maximize: $\lambda$
   
   subject to: $P \succcurlyeq 0$, $P \in M_{\vec{x}}$
   
   $\lambda \geq 0$
   
   $(\Delta^2 - \lambda \Delta)_t = (\vec{x} P \vec{x}^T)_t$, for all $t \in B_{2d}(e, S)$

4. Compute $Q = [\vec{q}_e, \ldots, \vec{q}_{gn}] \sim \sqrt{P}$
Action Plan 2

1. Pick $G = \langle S|R \rangle$;
2. Set $\mathbf{x} = (e, g_1, g_2, \ldots, g_n)$, $g_i \in B_d(e, S)$ ($d = 2, 3, \ldots$);
3. Solve the problem (numerically):
   \[
   \begin{align*}
   \text{maximize: } & \lambda \\
   \text{subject to: } & P \succ 0, \quad P \in \mathbb{M}_{\mathbf{x}} \\
   & \lambda \geq 0 \\
   & (\Delta^2 - \lambda \Delta)_t = (\mathbf{x}P\mathbf{x}^T)_t, \quad \text{for all } t \in B_{2d}(e, S)
   \end{align*}
   \]
4. Compute $Q = [\mathbf{q}_e, \ldots, \mathbf{q}_g]$ $\sim \sqrt{P}$
5. Setting $\xi_g = \langle \mathbf{x}, \mathbf{q}_g \rangle$ we have
   \[
   \Delta^2 - \lambda \Delta = \sum \xi_g^* \xi_g + r, \quad \text{where } r \in l[G] \text{ and } \|r\|_1 < \varepsilon.\]
Action Plan 2

1. Pick $G = \langle S | R \rangle$;
2. Set $\mathbf{x} = (e, g_1, g_2, \ldots, g_n), \quad g_i \in B_d(e, S) \ (d = 2, 3, \ldots)$;
3. Solve the problem (numerically):

   maximize: $\lambda$

   subject to: $P \succ 0, \quad P \in \mathbb{M}_x$

   $\lambda \geq 0$

   $(\Delta^2 - \lambda \Delta)_t = (\mathbf{x} P \mathbf{x}^T)_t, \quad$ for all $t \in B_{2d}(e, S)$

4. Compute $Q = [\mathbf{q}_e, \ldots, \mathbf{q}_{g_n}] \sim \sqrt{P}$

5. Setting $\xi_g = \langle \mathbf{x}, \mathbf{q}_g \rangle$ we have

   $\Delta^2 - \lambda \Delta = \sum \xi_g^* \xi_g + r, \quad$ where $r \in I[G]$ and $\|r\|_1 < \epsilon$.

6. Finally $\Delta^2 - (\lambda - 2^{d-1}\epsilon) \Delta = \sum \xi_j^* \xi_j + (r + 2^{d-1}\epsilon \Delta) \geq 0$, hence

   $\lambda_1(G, S) \geq (\lambda - 2^{d-1}\epsilon) \quad$ is certified.
Action Plan 2

1. Pick $G = \langle S | R \rangle$;
2. Set $\vec{x} = (e, g_1, g_2, \ldots, g_n)$, $g_i \in B_d(e, S)$ ($d = 2, 3, \ldots$);
3. Solve the problem (numerically):
   
   maximize: $\lambda$
   
   subject to: $P \succeq 0$, $P \in \mathbb{M}_{\vec{x}}$
   
   $\lambda \geq 0$
   
   $(\Delta^2 - \lambda \Delta)_t = (\vec{x}P\vec{x}^T)_t$, for all $t \in B_{2d}(e, S)$

4. Compute $Q = [\vec{q}_e, \ldots, \vec{q}_{gn}] \sim \sqrt{P}$
   
   $P \rightarrow \sqrt{P} \rightarrow \sqrt{P_{int}} \rightarrow \sqrt{P_{int}^{aug}} \rightarrow Q \in \mathbb{M}_{\vec{x}}(\mathbb{R}_{1F})$

5. Setting $\xi_g = \langle \vec{x}, \vec{q}_g \rangle$ we have
   
   $\Delta^2 - \lambda \Delta = \sum \xi_g^* \xi_g + r$, where $r \in l[G]$ and $\|r\|_1 < \varepsilon$.

6. Finally $\Delta^2 - (\lambda - 2^{d-1}\varepsilon)\Delta = \sum \xi_j^* \xi_j + (r + 2^{d-1}\varepsilon \Delta) \geq 0$, hence
   
   $\lambda_1(G, S) \geq (\lambda - 2^{d-1}\varepsilon)$ is certified.
CONCRETE EXAMPLES
\[ \sqrt{P} = Q \in M_{\overline{x}}, \text{ where } \overline{x} = B_2(e, E(3)), \text{ i.e. rows and columns are indexed by elements in } (SL(3, \mathbb{Z}), E(3)) \text{ of word length } \leq 2. \text{ In this case} \]

\[ \Delta^2 - 0.28\Delta = \sum_{i=1}^{121} \xi_i^* \xi_i + r, \quad ||r||_1 \in [3.8508, 3.8511] \cdot 10^{-7} \]
<table>
<thead>
<tr>
<th>$G$</th>
<th>$n$</th>
<th>$m$</th>
<th>$\lambda$</th>
<th>$|r|_1 &lt;$</th>
<th>$lb_\kappa$</th>
<th>$&lt; \kappa$</th>
<th>$ub_\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SL(3, \mathbb{Z})$</td>
<td>390,287</td>
<td>935,021</td>
<td>0.5405</td>
<td>$5.2 \cdot 10^{-7}$</td>
<td>0.19</td>
<td>0.30014</td>
<td>0.81650</td>
</tr>
<tr>
<td>$SL(4, \mathbb{Z})$</td>
<td>93,962</td>
<td>263,122</td>
<td>1.3150</td>
<td>$5.2 \cdot 10^{-8}$</td>
<td>0.00106</td>
<td>0.33103</td>
<td>0.70711</td>
</tr>
<tr>
<td>$SL(5, \mathbb{Z})$</td>
<td>628,882</td>
<td>1,757,466</td>
<td>2.6500</td>
<td>$2.0 \cdot 10^{-4}$</td>
<td>0.00105</td>
<td>0.36400</td>
<td>0.63246</td>
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</tbody>
</table>
### $\text{SAut}(F_4)$

<table>
<thead>
<tr>
<th></th>
<th>$n$</th>
<th>$m$</th>
<th>$\lambda$</th>
<th>$|r|_1 &lt;$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{SAut}(F_4)$</td>
<td>3,157,730</td>
<td>1,777,542</td>
<td>0.0100</td>
<td>7.4</td>
</tr>
</tbody>
</table>

(after weeks of computation)
$\text{SAut}(F_5)$

<table>
<thead>
<tr>
<th>$G$</th>
<th>$n$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{SAut}(F_5)$</td>
<td>21,538,881</td>
<td>11,154,301</td>
</tr>
</tbody>
</table>
Find a finite group $K \triangleleft \text{Aut} \left( \text{SAut}(F_n) \right)$ which keeps the generating set $S$ and (thus) $\Delta^2 - \lambda \Delta$ invariant ($K = \mathbb{Z}_2 \wr S_5$);
Find a finite group $K < \text{Aut}(\text{SAut}(F_n))$ which keeps the generating set $S$ and (thus) $\Delta^2 - \lambda \Delta$ invariant ($K = \mathbb{Z}_2 \wr S_5$);

The optimisation problem for $\lambda$ has a $K$-invariant solution.
Find a finite group $K < \text{Aut}(\text{SAut}(F_n))$ which keeps the generating set $S$ and (thus) $\Delta^2 - \lambda \Delta$ invariant ($K = \mathbb{Z}_2 \wr S_5$);

- The optimisation problem for $\lambda$ has a $K$-invariant solution
- Decompose $B_{2d}(e, S)$ into orbits of $K$ (7229 of them)
Find a finite group $K < \text{Aut}(\text{SAut}(F_n))$ which keeps the generating set $S$ and (thus) $\Delta^2 - \lambda \Delta$ invariant ($K = \mathbb{Z}_2 \wr S_5$);

The optimisation problem for $\lambda$ has a $K$-invariant solution

Decompose $B_{2d}(e, S)$ into orbits of $K$ (7229 of them)

Decompose $B_{d}(e, S)$ into irreducible representations of $K$
- Find a finite group $K \triangleleft \text{Aut}(\text{SAut}(F_n))$ which keeps the generating set $S$ and (thus) $\Delta^2 - \lambda \Delta$ invariant ($K = \mathbb{Z}_2 \wr S_5$);
- The optimisation problem for $\lambda$ has a $K$-invariant solution
- Decompose $B_{2d}(e, S)$ into orbits of $K$ (7229 of them)
- Decompose $B_d(e, S)$ into irreducible representations of $K$
- Using minimal projection system for $K$ reduce the size of the optimisation problem (29 semidefinite constraints, 13233 variables)
Find a finite group $K < \text{Aut}(\text{SAut}(F_n))$ which keeps the generating set $S$ and (thus) $\Delta^2 - \lambda \Delta$ invariant ($K = \mathbb{Z}_2 \wr S_5$);

The optimisation problem for $\lambda$ has a $K$-invariant solution

Decompose $B_{2d}(e, S)$ into orbits of $K$ (7229 of them)

Decompose $B_d(e, S)$ into irreducible representations of $K$

Using minimal projection system for $K$ reduce the size of the optimisation problem (29 semidefinite constraints, 13233 variables)

Solve the smaller problem and reconstruct the solution $P$ of the larger one.
$\text{SAut}(F_5)$

<table>
<thead>
<tr>
<th>$G$</th>
<th>$n$</th>
<th>$m$</th>
<th>$\lambda$</th>
<th>$|r|_1 &lt;$</th>
<th>$&lt; \kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{SAut}(F_5)$</td>
<td>13,233</td>
<td>7,230</td>
<td>1.3000</td>
<td>$2.1 \cdot 10^{-6}$</td>
<td>0.18028</td>
</tr>
</tbody>
</table>
Bibliography


M. Kaluba and P. Nowak, Certifying numerical estimates of spectral gaps, Groups, Complexity, Cryptology, arXiv: 1703.09680

M. Kaluba, P. Nowak and N. Ozawa, Aut($\mathbb{F}_5$) has property (T), arXiv: 1712.07167
[...] certain things first became clear to me by a mechanical method, although they had to be proved by geometry afterwards [...] But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge.
[...] certain things first became clear to me by a mechanical method, although they had to be proved by geometry afterwards [...] But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge.

(The Method of Mechanical Theorems, Archimedes)