How asymmetric are asymmetric manifolds?

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June 22, 2016

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Asymmetric manifolds

Definition

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There might be manifolds smoothly asymmetric which are not topologically asymmetric.

Innocent beginning

Theorem ((Borel, 1969?), Conner&Raymond, 1971)

Let G denote a finite subgroup of homeomorphisms of a closed, connected, aspherical manifold M. Consider the homomorphism

$$j\colon G \to \operatorname{Out}(\pi_1(M))$$

which sends $f \in G < \operatorname{Homeo}(M)$ to the outer automorphism of $\pi_1(M)$ induced by the homeomorphism f. If $\pi_1(M)$ has trivial center, then j is a monomorphism.

Theorem (Conner, Raymond and Weinberger, 1971)
Mapping toruses M_f of certain maps $f\colon T^n \to T^n$ are closed aspherical manifolds such that

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Theorem (Raymond, Tollefson, 1976)

There exists aspherical 3-manifold M with the outer automorphisms group of π_1 is torsion free. (i.e. $\mathrm{Homeo}(M)$ contain no finite subgroup).

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Question (Raymond & Schultz, 1976)

Does there exist a closed simply connected manifold on which no finite group act effectively? (A weaker question, no involution?)

[repeated in 2002 by Adem & Davis]

Sidenote: converse to Borel

Theorem (Malfait, 1998)
Borel conditions are also necessary for e.g. flat Riemannian manifolds, infra-nilmanifolds and infra-solvmanifolds of type (R).

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The existence of smooth simply connected manifolds with no finite symmetries is still an open problem.

Cohomology and asymmetry

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Theorem (Puppe, 1995)

($\mathrm{char}\, k = p$) *Let M be a compact manifold such that*

- $H^*(M; k)$ *has no non-trivial automorphism of order* p*,*
- $H^*(M; k)$ *has no non-trivial derivation of negative degree,*
- $H^*(M; k)$ *has no non-trivial deformation of negative weight, and*
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Then M does not admit any non-trivial action of $\mathbb{Z}/_p$ (in the case $p = 2$: orientation preserving action).

Theorem (Wall, 1966)
The diffeomorphism classes of elements of 6-dimensional, spin manifolds with torsion free cohomology generated in 2-nd degree correspond bijectively to isomorphism classes of (H, μ, p_1)*:*

1. *a free* $\mathbb{Z}$*-module* H *of finite rank, corresponding to* $H^2(M; \mathbb{Z})$*,*
2. *a trilinear, symmetric form* $\mu\colon H \times H \times H \to \mathbb{Z}$*, corresponding to the cup product in* $H^*(M; \mathbb{Z})$*,*
3. *a linear map* $p_1 \in \hom(H, \mathbb{Z})$*, corresponding to the dual of the first Pontrjagin class,*

subject to the following conditions:

(a) $\mu(x, x, y) \equiv \mu(x, y, y) \pmod 2$ *for* $x, y \in H$*,*

(b) $p_1(x) \equiv 4\mu(x, x, x) \pmod{24}$ *for* $x \in H$*.*

Set $H = \mathbb{Z}^6$ and $f\colon H \to \mathbb{Z}$,

$$\begin{aligned} f(x_1, \dots, x_6) = 6\Big(& x_1 x_4^2 - x_1^2 x_4 + x_2 x_4^2 + x_2 x_4^2 - x_2^2 x_5 + x_2 x_5^2 + \\ & + x_3^2 x_4 - x_3 x_4^2 + x_3^2 x_6 + x_3 x_6^2 + x_5^2 x_6 + x_5 x_6^2 + \\ & + x_1 x_2 x_4 + x_1 x_2 x_5 + x_1 x_3 x_6 + x_2 x_4 x_6 + x_3 x_5 x_6 + \\ & + x_4 x_5 x_6 + x_4 x_5 x_6 + x_4^3 + x_6^3 \Big). \end{aligned}$$

Then symmetric-trilinearisation of f provides a family $\mathcal{M}_{As}$ of almost asymmetric manifolds.

Product Actions

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$$\begin{aligned} G \times (M \times N) &\longrightarrow M \times N \\ (g, (x, y)) &\longmapsto \begin{bmatrix} \varphi(g) & 0 \\ 0 & \psi(g) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = [\varphi(g)x, \psi(g)y] \end{aligned}$$

Where φ and ψ denote actions of G on maifolds M, N respectively.

Question

Given a product of manifolds $M \times N$ what are possible actions on the space?

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When there are plenty of actions on both M and N, we tend to believe that some of them might be interweaved to create a non-product one.

Choose M with as few symmetries as possible – an asymmetric one.
The most symmetric choice for N is a sphere.

Question

What is the minimal n (depending on M and G) such that there exist a non-product action of G on $M \times S^n$?

Outline

In this talk the we will focus on cases:

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joint work with **Zbigniew Błaszczyk**

Actions on $M \times S^2$ are exotic

Non-product actions

General assumptions:

- $G \cong \mathbb{Z}/_p$, a finite cyclic group or $G \cong S^1$;
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Proposition

There exist effective, non-product actions of G on $M \times S^2$.

If M is smooth, then the action can be arranged to be smooth as well.

Codimension-2 fixed point sets of G-actions

Proposition

Let X be a contractible, $(m+1)$-dimensional ($m \geq 3$) manifold with smooth boundary $\partial X = \Sigma$ (Σ is necessarily a $\mathbb{Z}$-homology sphere). Then there exist effective, smooth G-action on sphere S^{m+2} with the fixed-point set diffeomorphic to Σ.

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- The action restricted to the boundary is the desired one.

□

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- There exists a smooth action of G on S^{m+2} with the fixed point set diffeomorphic to Σ and tangential G-module at Σ isomorphic to $V \oplus m\mathbf{1}_G$.
- Form a G-connected sum

$$M \times S(V \oplus \mathbb{R}) \# S^{m+2} \cong M \times S^2.$$

Proof: (continued)

- None of these allows for two components

$$M \sqcup M\#\Sigma$$

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Proposition

There exist exotic orientation preserving involutions on $M \times S^1$ for any asymmetric M of dimension $m \geq 4$.

Free S^1-actions on $M^6 \times S^1$ are standard

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All orientation preserving, free S^1-actions on $M^6 \times S^1$ are equivalent to the product actions.

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We strongly believe that the following is also true:

Conjecture

All free $\mathbb{Z}/_p$-actions on $M^6 \times S^1$ are equivalent to a product action ($p \neq 2$).

Proof:

All S^1-bundles are determined by their first Chern class

$$c_1(\xi) = c(\xi)^*(x),$$

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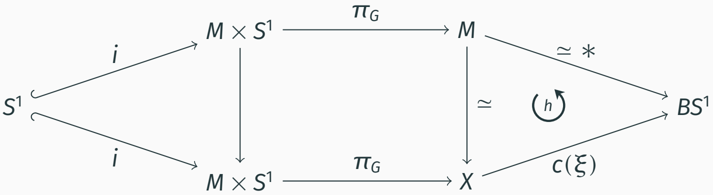
Assume so for now.

Proof: (continued)

Then we have a commuting diagram:



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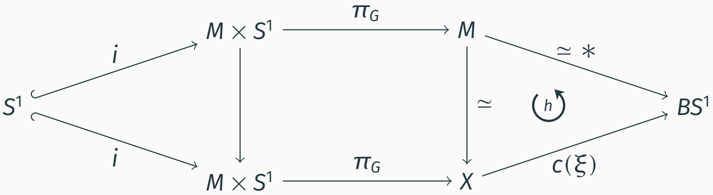

So we know that over (a manifold) X the trivial S^1-bundle satisfies

$$M \times S^1 \simeq X \times S^1.$$

Proof: (continued)

Observe that this gives us just a homotopy equivalence

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- We already have $M \times S^1 \cong X \times S^1$. Lift it to

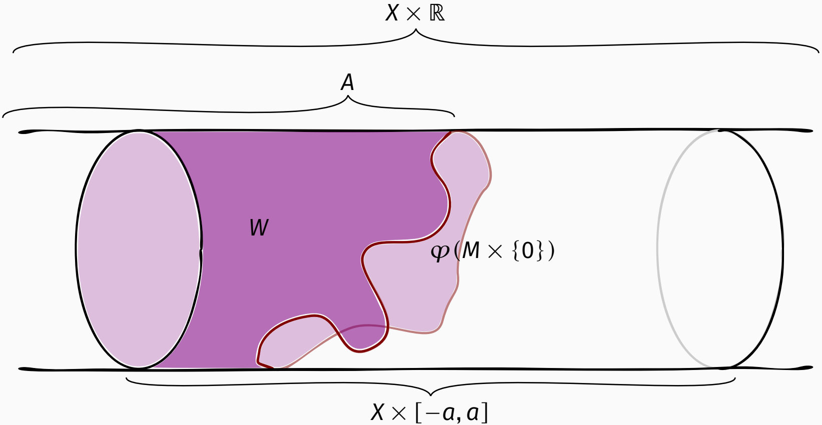
$$\varphi : M \times \mathbb{R} \to X \times \mathbb{R}.$$

- Image of $\varphi(M \times \{0\})$ belongs to $X \times (-a, a)$ for some $a > 0$. Set A for the connected component of $(X \times \mathbb{R}) - \varphi(M \times \{0\})$ such that

$$W = A \cap (X \times (-\infty, a])$$

  is non empty.

Proof: (continued)



- W is an h-cobordism between X and $\varphi(M)$ which yields a diffeomorphism $M \to X$. $\square$

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Proof of this fact relays on:

Fact: Multiplication by $c_1(\xi)$ can be identified with a differential on the first non-trivial page of the Leray-Serre spectral sequence of the fibration.

Triviality of the first Chern class

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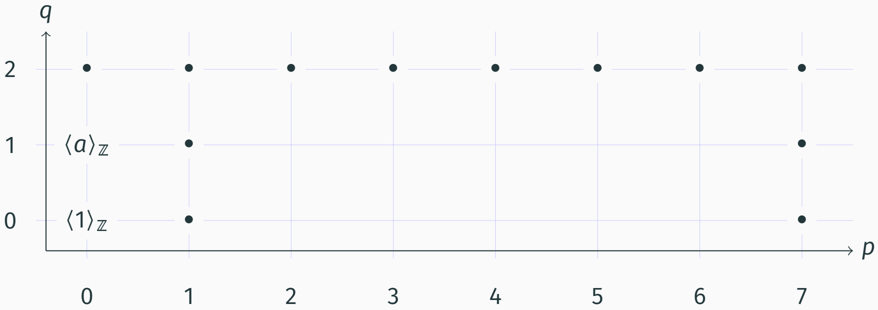
If S^1 acts preserving orientation, $\pi_1(X)$ acts trivially on $H^*(S^1)$ and we have Serre spectral sequence

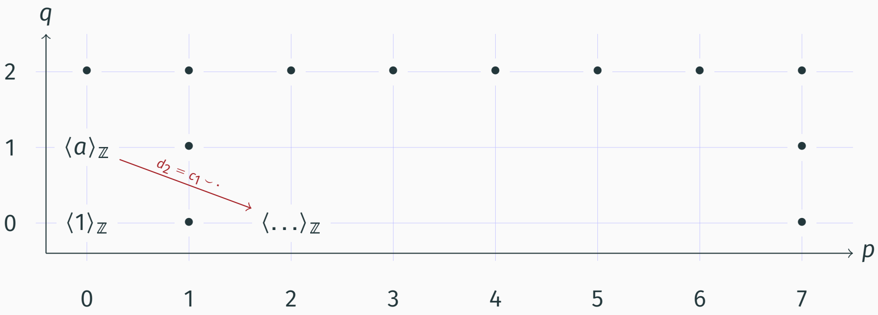
$$E_2^{p,q} = H^p\left(X, H^q(S^1;\mathbb{Z})\right) \Rightarrow H^{p+q}(M \times S^1;\mathbb{Z})$$

with untwisted coefficients.



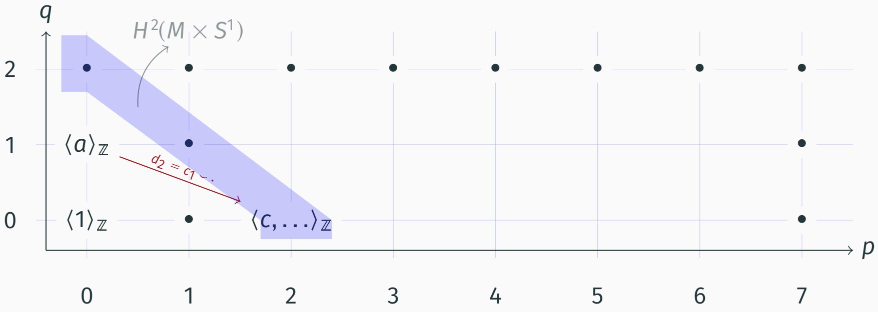
q
2
1
0
⟨a⟩ℤ
⟨1⟩ℤ
p
0
1
2
3
4
5
6
7

- $d_2 \colon E_2^{0,1} \to E_2^{2,0}$ is multiplication by $c_1(\xi)$



- $d_2 \colon E_2^{0,1} \to E_2^{2,0}$ is multiplication by $c_1(\xi)$
- Set $d_2(a) = c$. We claim that $c \in \mathbb{Z}/_k$ is a generator.

$c \in \mathrm{Tor}(H^2(X)) = H_1(X) = \mathbb{Z}/_k$



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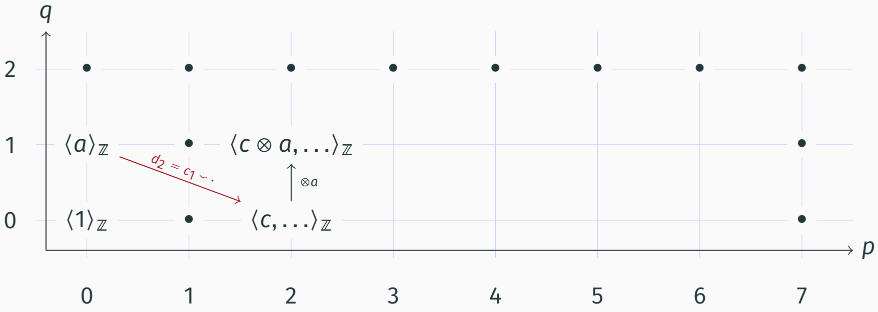

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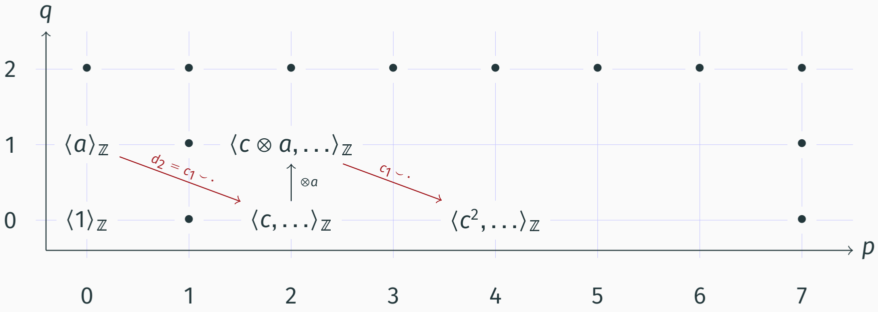

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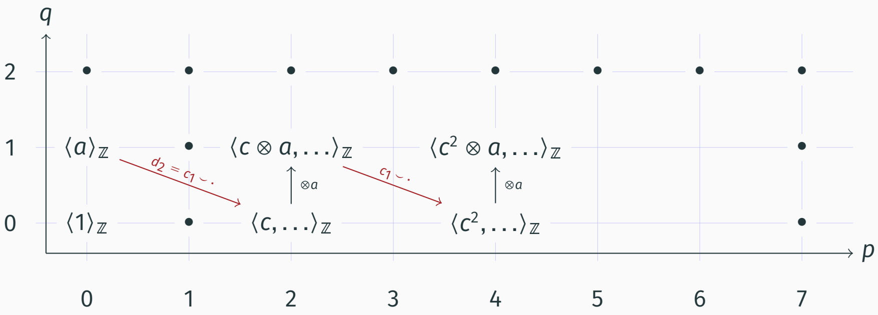

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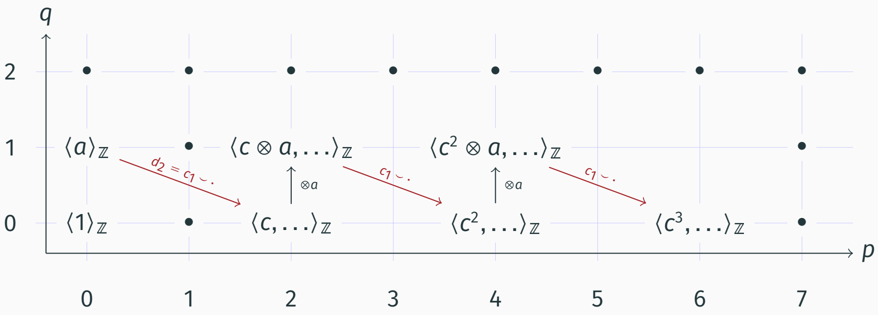

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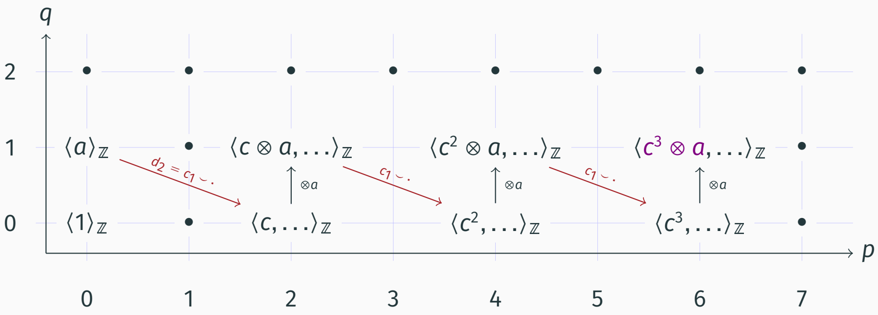

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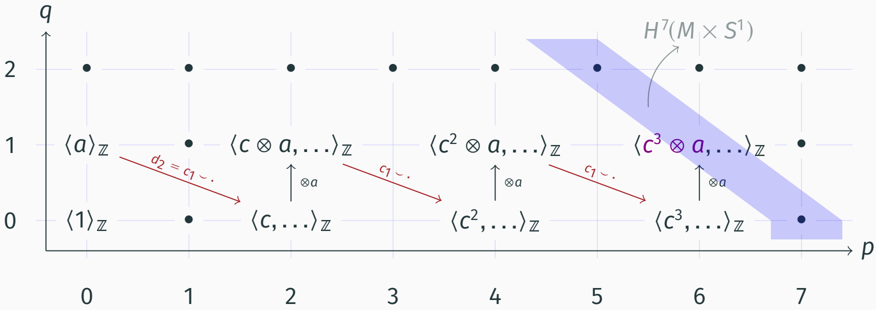

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- $c^3 \otimes a$ survives to E_∞ and hence to $H^7(M \times S^1)$.
- But $H^7(M \times S^1) = \mathbb{Z}$, so $d_2(c^2 \otimes a) = c^3 = 0$.

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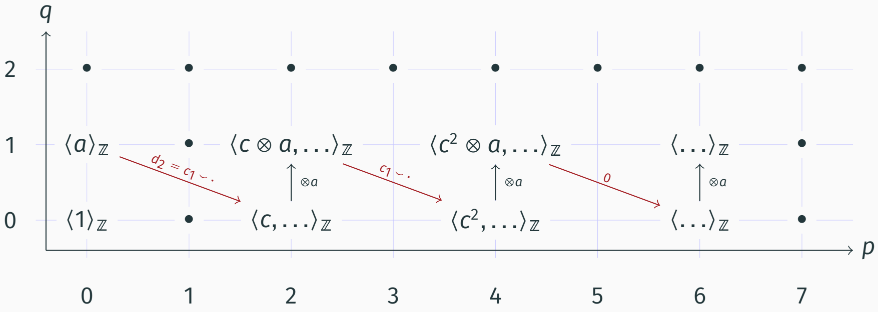
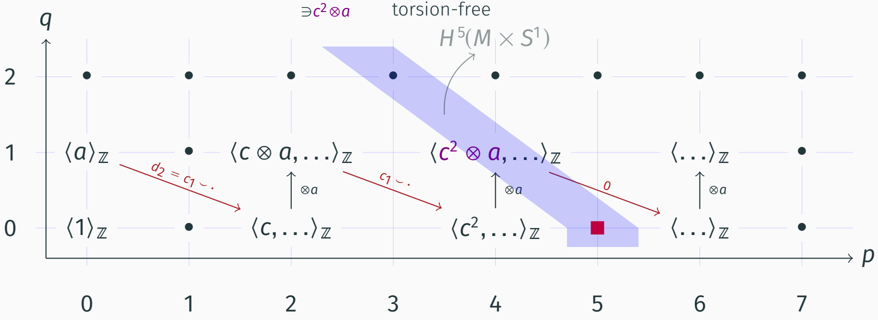


- $d_2\colon E_2^{0,1} \to E_2^{2,0}$ is multiplication by $c_1(\xi)$
- Set $d_2(a) = c$. We claim that $c \in \mathbb{Z}/_k$ is a generator.
- Since $d_2(c \otimes a) = c^2$, push $c \rightsquigarrow c^3 \otimes a \in E_2^{6,1}$.
- $c^3 \otimes a$ survives to E_∞ and hence to $H^7(M \times S^1)$.
- But $H^7(M \times S^1) = \mathbb{Z}$, so $d_2(c^2 \otimes a) = c^3 = 0$.
- Now $c^2 \otimes a$ survives to E_∞, so we have an extension

$c \in \mathrm{Tor}(H^2(X)) = H_1(X) = \mathbb{Z}/_k$

$$0 \to \underbrace{E_2^{4,1}}_{\ni c^2 \otimes a} \hookrightarrow \underbrace{H^5(M \times S^1)}_{\text{torsion-free}} \to \blacksquare \to 0.$$



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It also suggests, that the fact is more general and it holds for all manifolds with torsion-free cohomology in even degrees.

Free actions on $M \times S^1$

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Theorem

Free $\mathbb{Z}/_p$-actions on $M^6 \times S^1$ are smoothly/topologically conjugated if and only if their orbit spaces are homeo-/diffeomorphic (p prime).

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Proof: (for $p = 2$)

Let τ_1, τ_2 be two involutions on $M \times S^1$. Suppose

$$f\colon\ (M \times S^1, q_1)/\tau_1 \to (M \times S^1, q_2)/\tau_2$$

is a homeomorphism.

Proof: (for $p = 2$)

$$\begin{array}{ccc} (M \times S^1, x_0) & & (M \times S^1, x_1) \\ \Big\downarrow {\scriptstyle p_1} & & \Big\downarrow {\scriptstyle p_2} \\ (M \times S^1, q_1)/\tau_1 & \xrightarrow{\quad f \quad} & (M \times S^1, q_2)/\tau_2 \end{array}$$

Proof: (for $p = 2$)

$$
\begin{array}{ccc}
(M \times S^1, x_0) & \overset{F}{\dashrightarrow} & (M \times S^1, x_1) \\
{\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\
(M \times S^1, q_1)/\tau_1 & \overset{f}{\longrightarrow} & (M \times S^1, q_2)/\tau_2
\end{array}
$$

The lift F of f exists if and only if

$$f_* \circ (p_1)_*(\pi_1) \subset (p_2)_*(\pi_1).$$

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This is always the case if e.g. $\pi_1\left((M \times S^1)/\tau_i\right) \cong \mathbb{Z}$.

Proof: (end)

Then $\tau_2 \circ F$ and $F \circ \tau_1$ are lifts of f, both distinct from F.

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$$\tau_2 = F \circ \tau_1 \circ F^{-1}$$

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which ends the proof. □

Lemma

Suppose that a finite group acts freely on $M \times S^1$, Then $\pi_1\left((M \times S^1)/G\right) \cong \mathbb{Z}$.

Proof

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- Let G act freely on $M \times S^1$, and set $\pi = \pi_1\left((M \times S^1)/G\right)$. Then

$$0 \to \mathbb{Z} \to \pi \to G \to 0.$$

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- $(M \times S^1)/G$ is still universally covered by $M \times \mathbb{R}$, therefore π acts (as deck transformations) on $M \times \mathbb{R}$.
- Yet we claim that no finite group acts freely on $M \times \mathbb{R}$, thus $\pi \cong \mathbb{Z}$.

$\square$

Claim

No finite group acts freely on $M \times \mathbb{R}$.

Proof. Consider the fibration $M \times \mathbb{R} \to (M \times \mathbb{R})/\mathbb{Z}/_p \to K(\mathbb{Z}/_p, 1)$ and its associated Serre spectral sequence

$$E_2^{s,t} \cong H^s\Big(K(\mathbb{Z}/_p, 1); \mathcal{H}^t(M \times \mathbb{R}; \mathbb{Z})\Big) \Longrightarrow H^{s+t}\Big((M \times \mathbb{R})/\mathbb{Z}/_p; \mathbb{Z}\Big).$$

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o. reversing Draw the E_2-page and take the plunge...

Plunge for $p = 2$.

- The ring $H^*(M;\mathbb{Z})$ admits a unique orientation reversing involution: $(+1)$ on H^0 and H^4, and (-1) on H^2 and H^6.

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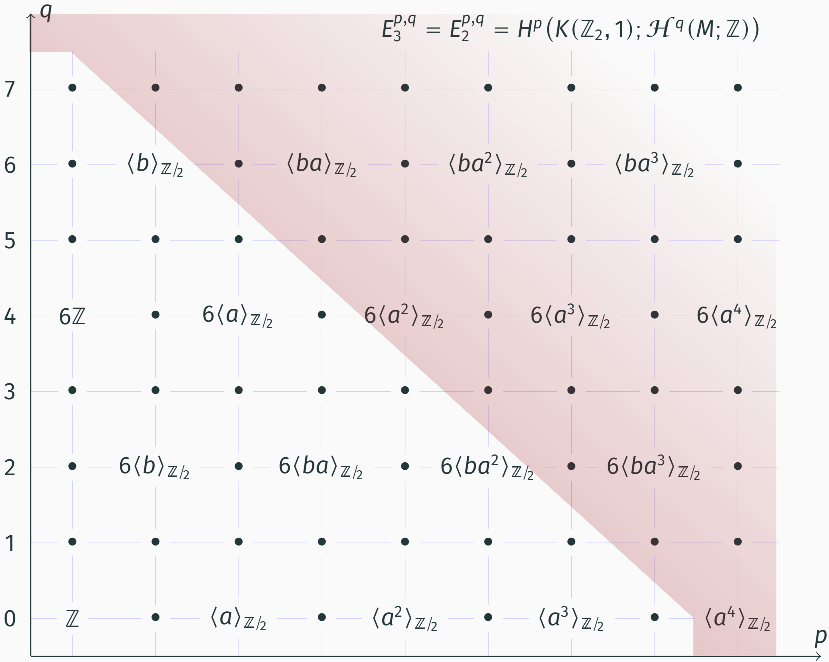
- it follows that

$$E_2^{*,6} \cong H^*\left(K(\mathbb{Z}/2,1), \widetilde{\mathbb{Z}}\right) \cong \begin{cases} \mathbb{Z}/2, & \text{for } * \text{ odd,} \\ 0, & \text{otherwise,} \end{cases}$$

  with the $E^{*,0}$-module structure given as $ax_{2i-1} = x_{2i+1}$ for x_{2i-1} the generator of $E_2^{2i-1,6}$. We will denote $x_1 = b$ and $x_{2i+1} = ba^i$



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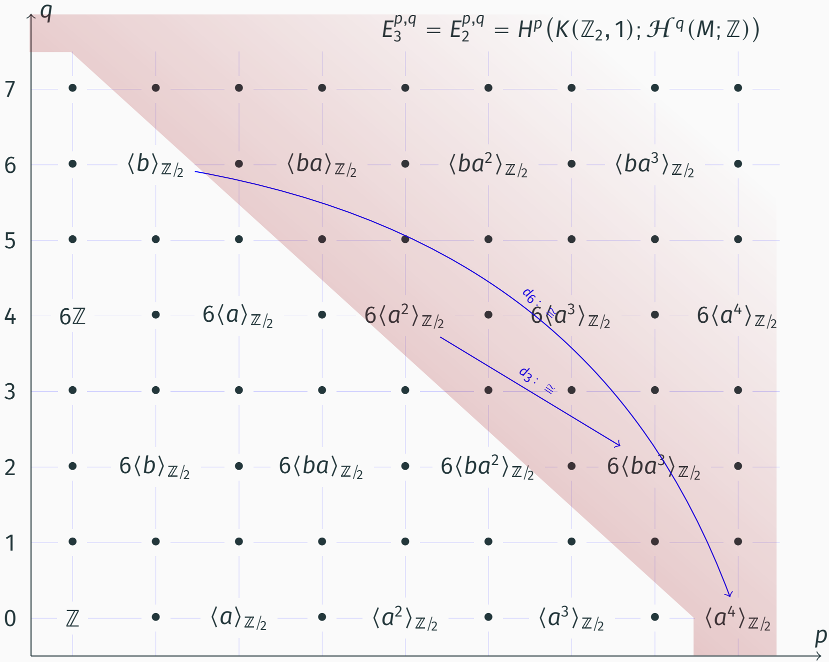

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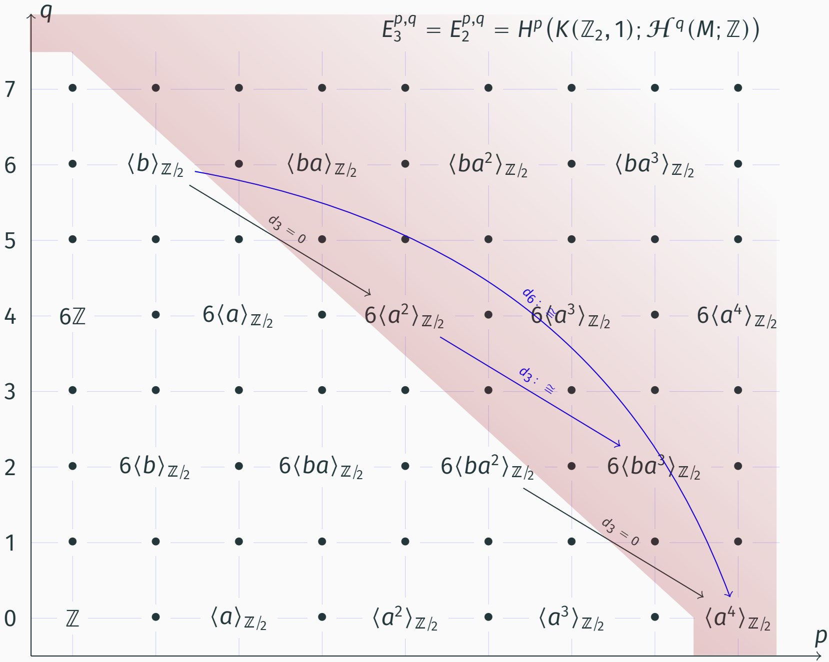

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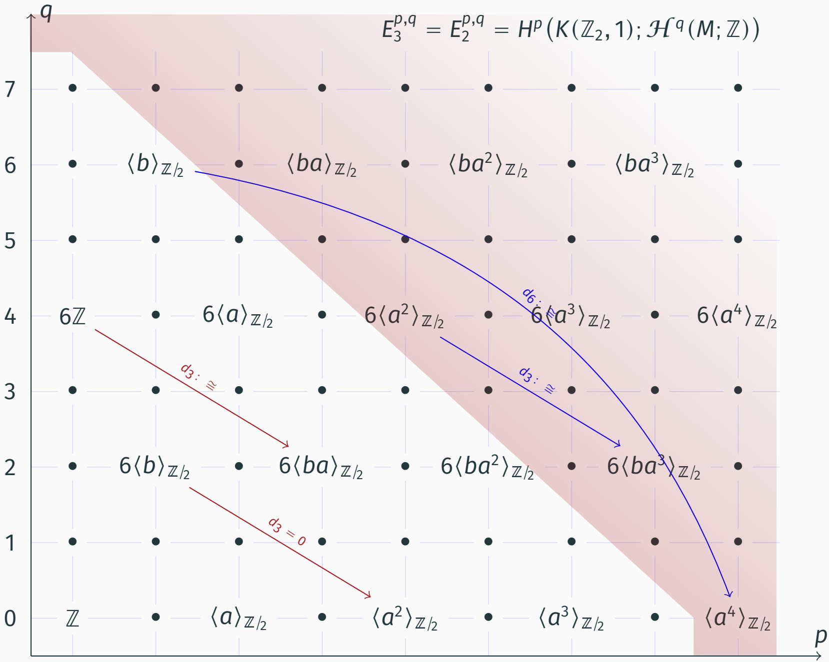
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Topological/smooth classification of the orbit spaces.

Open Questions

Let G be an arbitrary finite group and let N be the smallest dimension of faithful, irreducible representation of G.

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Problem

What are possible free actions $M \times S^1$, where M is asymmetric, aspherical manifold?

Thank You